JACOB'S LADDERS AND NEW CLASS OF INTEGRALS CONTAINING PRODUCT OF FACTORS ζ^2

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Abstract. In this paper we obtain new properties of a signal generated by the Riemann zeta-function on the critical line. At the same time we obtain an asymptotic formula for a new class of transcendental integrals connected with the Riemann zeta-function

1. Introduction

1.1. In the paper [4] we have obtained the following formula

$$\begin{aligned} &\frac{1}{U}\int_T^{T+U}\prod_{k=0}^n\left|\zeta\left(\frac{1}{2}+i\varphi_1^k(t)\right)\right|^2\sim \\ &\sim \prod_{k=0}^n \frac{1}{\varphi_1^k(T+U)-\varphi_1^k(T)}\int_{\varphi_1^k(T)}^{\varphi_1^k(T+U)}\left|\zeta\left(\frac{1}{2}+it\right)\right|^2 \mathrm{d}t, \\ &U\in\left(0,\frac{T}{\ln^2 T}\right],\ T\to\infty. \end{aligned} \tag{1.1}$$

A motivation for this formula was the well-known multiplicative formula

$$M\left(\prod_{k=1}^n X_k\right)=\prod_{k=1}^n M(X_k)$$

from the theory of probability where X_k are the independent random variables and M is the population mean. Some new art of the asymptotic independence of the partial functions

$$\left|\zeta\left(\frac{1}{2}+it\right)\right|^2,\ t\in\left[\varphi_1^k(T),\varphi_1^k(T+U)\right],\ k=0,1,\dots,n$$

is expressed by this formula.

1.2. For example, by using the mean-value theorem in (1.1) we obtain

$$\begin{aligned} &\left|\zeta\left(\frac{1}{2}+i\varphi_1^n(\bar{t}_n)\right)\right|^2\frac{1}{U}\int_T^{T+U}\prod_{k=0}^{n-1}\left|\zeta\left(\frac{1}{2}+i\varphi_1^k(t)\right)\right|^2\mathrm{d}t\sim \\ &\sim\frac{1}{\varphi_1^n(T+U)-\varphi_1^n(T)}\int_{\varphi_1^n(T)}^{\varphi_1^n(T+U)}\left|\zeta\left(\frac{1}{2}+it\right)\right|^2\mathrm{d}t\times \\ &\times\prod_{k=0}^{n-1}\frac{1}{\varphi_1^k(T+U)-\varphi_1^k(T)}\int_{\varphi_1^k(T)}^{\varphi_1^k(T+U)}\left|\zeta\left(\frac{1}{2}+it\right)\right|^2\mathrm{d}t \end{aligned}$$

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i. e. (see (1.1), $n\mapsto n-1$)

$$\left|\zeta\left(\frac{1}{2}+i\varphi_1^n(\bar{t}_n)\right)\right|^2\sim\frac{1}{\varphi_1^n(T+U)-\varphi_1^n(T)}\int_{\varphi_1^n(T)}^{\varphi_1^n(T+U)}\left|\zeta\left(\frac{1}{2}+it\right)\right|^2\mathrm{d}t. \tag{1.2}$$

But

$$\left|\zeta\left(\frac{1}{2}+i\varphi_1^n(\bar{t}_n)\right)\right|^2,\ \bar{t}_n\in(T,T+U)$$

is the mean value with respect to the set of functions

$$\left\{\left|\zeta\left(\frac{1}{2}+i\varphi_1^0(t)\right)\right|^2,\dots,\left|\zeta\left(\frac{1}{2}+i\varphi_1^{n-1}(t)\right)\right|^2\right\} \tag{1.3}$$

i. e. $\bar{t}_n$ is the nonlinear functional

$$\bar{t}_n=f_n[\varphi_1^1,\dots,\varphi_1^{n-1}];\ \varphi_1^0(t)=t \tag{1.4}$$

that is defined on the continuum set of points

$$(\varphi_1^1,\dots,\varphi_1^{n-1}). \tag{1.5}$$

Let us remind that there is the continuum set of the Jacob's ladders (see [1] generating the set of the iterations (1.5)). At the same time it follows from (1.2) that

$$\left|\zeta\left(\frac{1}{2}+i\varphi_1^n(t_n)\right)\right|^2\sim\left|\zeta\left(\frac{1}{2}+i\varphi_1^n(\tau)\right)\right|^2,\ \tau\in(T,T+U) \tag{1.6}$$

where τ is completely independent on the set of points (1.5).

Remark 1. Thus, the mean-value (1.2) with respect to the set of functions (1.3) is asymptotically independent on this set.

Remark 2. Now, let $k:\ 1<k<n$. Then we have the mean-value

$$\left|\zeta\left(\frac{1}{2}+i\varphi_1^k(\bar{t}_k)\right)\right|^2 \tag{1.7}$$

of the inner factor of the product in (1.1) with respect to the set (comp. (1.3))

$$\begin{aligned}&\left\{\left|\zeta\left(\frac{1}{2}+i\varphi_1^0(t)\right)\right|^2,\dots,\left|\zeta\left(\frac{1}{2}+i\varphi_1^{k-1}(t)\right)\right|^2,\left|\zeta\left(\frac{1}{2}+i\varphi_1^{k+1}(t)\right)\right|^2\dots\right.\\&\left.\dots,\left|\zeta\left(\frac{1}{2}+i\varphi_1^n(t)\right)\right|^2\right\},\end{aligned} \tag{1.8}$$

where (comp. (1.4))

$$\bar{t}_k=g_k[\varphi_1^1,\dots,\varphi_1^{k-1},\varphi_1^{k+1},\dots,\varphi_1^n]$$

is the functional defined on the continuum set of points

$$\left(\varphi_1^1,\dots,\varphi_1^{k-1},\varphi_1^{k+1},\dots,\varphi_1^n\right).$$

In this case the mean-value (1.7) is not, probable, asymptotically independent on the set (1.8).

1.3. In this paper we obtain new properties of the signal

$$Z(t)=e^{i\vartheta(t)}\zeta\left(\frac{1}{2}+it\right),\ \vartheta(t)=-\frac{t}{2}\ln\pi+\operatorname{Im}\ln\Gamma\left(\frac{1}{4}+i\frac{t}{2}\right)$$

generated by the Riemann zeta-function on the critical line. Namely, we obtain an asymptotic formula for a new class of transcendental integrals of the type

$$\int_T^{T+U}F[\varphi_1^{n+1}(t)]\prod_{k=0}^n\left|\zeta\left(\frac{1}{2}+i\varphi_1^k(t)\right)\right|^2\mathrm{d}t,\ U\in\left(0,\frac{T}{\ln^2T}\right],$$

where

$$F(w),\ w\in[\varphi_1^{n+1}(T),\varphi_1^{n+1}(T+U)],\ F(w)\geq 0\ (\leq 0)$$

is an arbitrary Lebesgue integrable function. In this direction, we obtain, for example, new asymptotic formulae generalizing our formulae containing the factors

$$\left|\zeta\left(\frac{1}{2}+i\varphi_1(t)\right)\right|^4,\ \left\{\arg\zeta\left(\frac{1}{2}+i\varphi_1(t)\right)\right\}^{2l},\ l\in\mathbb{N}.$$

Further we obtain the new effect for the macroscopic domains, i. e. for

$$U\in\left[T^{\frac{1}{3}+\epsilon},\frac{T}{\ln^2T}\right].$$

Namely:

(a) the transformations

$$[T,T+U]\to[\varphi_1^k(T),\varphi_1^k(T+U)],\ k=1,\dots,n+1$$

asymptotically preserve the measure (the length) of the segment $[T,T+U]$, i. e. that

$$|[\varphi_1^k(T),\varphi_1^k(T+U)]|\sim U,\ k=1,\dots,n+1,\ T\to\infty;$$

(b) the segments

$$\{[\varphi_1^k(T),\varphi_1^k(T+U)]\}_{k=0}^{n+1}$$

are distributed with an exact asymptotic regularity.

2. The result

2.1. Let us remind that the formula (comp. [4], (3.7), (3.8))

$$\tilde{Z}^2(t)=\frac{\mathrm{d}\varphi_1(t)}{\mathrm{d}t},\ \varphi_1(t)=\frac{1}{2}\varphi(t),\ t\geq T_0[\varphi] \tag{2.1}$$

and

$$\tilde{Z}^2(t)=\frac{Z^2(t)}{2\Phi'_\varphi[\varphi(t)]}=\frac{\left|\zeta\left(\frac{1}{2}+it\right)\right|^2}{\left\{1+\mathcal{O}\left(\frac{\ln\ln t}{\ln t}\right)\right\}\ln t} \tag{2.2}$$

where $\varphi(t)$ is the Jacob's ladder, i. e. the exact solution of the nonlinear integral equation

$$\int_0^{\mu[x(T)]}Z^2(t)e^{-\frac{2}{x(T)}}\mathrm{d}t=\int_0^T Z^2(t)\mathrm{d}t, \tag{2.3}$$
$$\mu(y)\geq 7y\ln y,\ \mu(y)\to y=\varphi_\mu(T)=\varphi(T)$$

(see [1]). Next, we have (see [4], (2.1))

$$y=\frac{1}{2}\varphi(t)=\varphi_1(t);\ \varphi_1^0(t)=t,\ \varphi_1^1(t)=\varphi_1(t),$$
$$\varphi_1^2(t)=\varphi_1(\varphi_1(t)),\dots,\varphi_1^k(t)=\varphi_1(\varphi_1^{k-1}(t)),\dots$$

where $\varphi_1^k(t)$ stands for the kth iteration of the Jacob's ladder

$$y=\varphi_1(t)$$

(of course, $\varphi_1^k(t),\ t\in[T,T+U]$ are the increasing functions.) The following Theorem holds true.

Theorem. Let

$$U\in\left(0,\frac{T}{\ln^2 T}\right]. \tag{2.4}$$

Then for every fixed $n\in\mathbb{N}$ and for every Lebesgue-integrable function

$$F(t),\ t\in[\varphi_1^{n+1}(T),\varphi_1^{n+1}(T+U)],\quad F(t)\geq 0\ (\leq 0)$$

we have

$$\begin{split}&\int_T^{T+U}F[\varphi_1^{n+1}(t)]\prod_{k=0}^n\left|\zeta\left(\frac{1}{2}+i\varphi_1^k(t)\right)\right|^2\mathrm{d}t\sim\\ &\sim\left\{\int_{\varphi_1^{n+1}(T)}^{\varphi_1^{n+1}(T+U)}F(t)\mathrm{d}t\right\}\ln^{n+1}T,\ T\to\infty\end{split} \tag{2.5}$$

where

$$\varphi_1^k(T+U)-\varphi_1^k(T)<\frac{1}{2n+3}\frac{T}{\ln T},\ k=1,\dots,n+1, \tag{2.6}$$

$$\varphi_1^k(T+U)-\varphi_1^k(T)>0.2\times\frac{T}{\ln T},\ k=0,1,\dots,n. \tag{2.7}$$

Next, in the macroscopic case (comp. (2.4))

$$U\in\left[T^{\frac{1}{3}+\epsilon},\frac{T}{\ln^2 T}\right], \tag{2.8}$$

we have more exact information

$$|[\varphi_1^k(T),\varphi_1^k(T+U)]|=\varphi_1^k(T+U)-\varphi_1^k(T)\sim U,\ k=1,\dots,n, \tag{2.9}$$

$$\varphi_1^k(T)-\varphi_1^{k+1}(T+U)\sim(1-c)\frac{T}{\ln T},\ k=0,1,\dots,n, \tag{2.10}$$

$$\begin{split}&\rho\{[\varphi_1^{k-1}(T),\varphi_1^{k-1}(T+U)];[\varphi_1^k(T),\varphi_1^k(T+U)]\}\sim(1-c)\frac{T}{\ln T},\\ &k=1,\dots,n+1\end{split} \tag{2.11}$$

where ρ denotes the distance of the corresponding segments.

Remark 3. In the macroscopic case (2.8) the following is true. The system of iterated segments

$$[\varphi_1^{n+1}(T),\varphi_1^{n+1}(T+U)],[\varphi_1^n(T),\varphi_1^n(T+U)],\dots,[T,T+U] \tag{2.12}$$

is the disconnected set and its components are:

(a) asymptotically equal (see (2.9)),

(b) distributed with the asymptotic regularity from the right to the left (see (2.10) – (2.12)).

Remark 4. Every Jacob's ladder

$$\varphi_1(t)=\frac{1}{2}\varphi(t)$$

(see (2.1)) where $\varphi(t)$ is the exact solution of the nonlinear integral equation (2.3) is the asymptotic solution of the following nonlinear integro-iteration equation

$$\begin{aligned}&\frac{1}{U}\int_T^{T+U}F[x^{n+1}(t)]\prod_{k=0}^n\left|\zeta\left(\frac{1}{2}+ix^k(t)\right)\right|^2\mathrm{d}t= \\ &=\left\{\int_{x^{n+1}(T)}^{x^{n+1}(T+U)}F(t)\mathrm{d}t\right\}\ln^{n+1}T\end{aligned} \tag{2.13}$$

(comp. (2.5)) where

$$x_0(t)=t,\ x^1(t)=x(t),\ x^2(t)=x(x(t)),\dots$$

i. e. the function $x^k(t)$ is the kth iteration of the function $x(t)$, (comp. (2.13) with [3], (11.1), (11.4), (11.6), (11.8), [4], (2.5) and [5], (2.6)).

Remark 5. Similar remarks like Remark 1 – Remark 2 hold true also when speaking *on the independence of the mean-value*.

2.2. By (2.9) and the formula (see [4], (3.5))

$$t-\varphi_1^{n+1}(t)\sim(1-c)(n+1)\frac{t}{\ln t}$$

we obtain easily (for example) from (2.5) the following

Corollary. In the macroscopic case (2.8) we have

$$\int_T^{T+U}\prod_{k=0}^n\left|\zeta\left(\frac{1}{2}+i\varphi_1^k(t)\right)\right|^2\mathrm{d}t\sim U\ln^{n+1}T,\ T\to\infty, \tag{2.14}$$

$$\begin{aligned}&\int_T^{T+U}\left|\zeta\left(\frac{1}{2}+i\varphi_1^{n+1}(t)\right)\right|^4\prod_{k=0}^n\left|\zeta\left(\frac{1}{2}+i\varphi_1^k(t)\right)\right|^2\mathrm{d}t\sim \\ &\sim\frac{1}{2\pi^2}U_1\ln^{n+5}T,\ U_1=T^{\frac{7}{8}+\epsilon},\end{aligned} \tag{2.15}$$

$$\begin{aligned}&\int_T^{T+U}\left\{\arg\zeta\left(\frac{1}{2}+i\varphi_1^{n+1}(t)\right)\right\}^{2l}\prod_{k=0}^n\left|\zeta\left(\frac{1}{2}+i\varphi_1^k(t)\right)\right|^2\mathrm{d}t\sim \\ &\sim\frac{(2l)!}{l!4^l}U\ln^{n+1}T(\ln\ln T)^l,\ U\in\left[T^{\frac{1}{2}+\epsilon},\frac{T}{\ln^2T}\right],\end{aligned} \tag{2.16}$$

$$\begin{aligned}&\int_T^{T+U}\left\{S_1\left[\varphi_1^{n+1}(t)\right]\right\}^{2l}\prod_{k=0}^n\left|\zeta\left(\frac{1}{2}+i\varphi_1^k(t)\right)\right|^2\mathrm{d}t\sim \\ &\sim d_lU\ln^{n+1}T,\ U\in\left[T^{\frac{1}{2}+\epsilon},\frac{T}{\ln^2T}\right],\end{aligned} \tag{2.17}$$

for every fixed $l,n\in\mathbb{N}$ where

$$S_1(T)=\int_0^TS(t)\mathrm{d}t,\ S(t)=\frac{1}{\pi}\arg\zeta\left(\frac{1}{2}+it\right),$$

and the argument is defined by the usual way (comp. [6], p. 179).

Remark 6. The formulae (2.15) – (2.17) are can be understand as generalization of our formulae [3], (8.3), [5], Lemma 2, (5.4), (5.5). The formula (2.14) can be compared with the formula (2.3) from the paper of reference [4] in the macroscopic case. The small improvements of the Heath-Brown exponent $\frac{7}{8}$ in (2.15) are irrelevant for our purpose.

3. Proof of Theorem

3.1. By the formula (see [4], (3.9))

$$\prod_{k=0}^{n}\tilde{Z}^2[\varphi_1^k(t)]=\frac{\mathrm{d}\varphi_1^{n+1}}{\mathrm{d}t}$$

we obtain

$$\begin{aligned}&\int_T^{T+U}F[\varphi_1^{n+1}(t)]\prod_{k=0}^{n}\tilde{Z}^2[\varphi_1^k(t)]\mathrm{d}t=\\ &=\int_T^{T+U}F[\varphi_1^{n+1}(t)]\mathrm{d}\varphi_1^{n+1}(t)=\int_{\varphi_1^{n+1}(T)}^{\varphi_1^{n+1}(T+U)}F(t)\mathrm{d}t,\end{aligned}$$

i. e.

$$\int_T^{T+U}F[\varphi_1^{n+1}(t)]\prod_{k=0}^{n}\tilde{Z}^2[\varphi_1^k(t)]\mathrm{d}t=\int_{\varphi_1^{n+1}(T)}^{\varphi_1^{n+1}(T+U)}F(t)\mathrm{d}t. \tag{3.1}$$

Since (see [4], (3.3), (3.6))

$$\begin{aligned}&t>\varphi_1^1(t)>\varphi_1^2(t)>\cdots>\varphi_1^{n+1}(t),\\ &(1-\epsilon)T<\varphi_1^{n+1}(T)<T\end{aligned}$$

then

$$(1-\epsilon)T<\varphi_1^{n+1}(T)<T+U,\ U\in\left(0,\frac{T}{\ln^2 T}\right].$$

Consequently,

$$T'\in(\varphi_1^{n+1}(T),T+U)\ \Rightarrow\ \ln T'=\ln T+\mathcal{O}(1). \tag{3.2}$$

Now, if we use the mean-value theorem on the left-hand side of (3.1) we obtain (see (2.2), (3.2))

$$\begin{aligned}&\int_T^{T+U}F[\varphi_1^{n+1}(t)]\prod_{k=0}^{n}\tilde{Z}^2[\varphi_1^k(t)]\mathrm{d}t\sim\\ &\sim\frac{1}{\ln^{n+1}T}\int_T^{T+U}\prod_{k=0}^{n}\left|\zeta\left(\frac12+i\varphi_1^k(t)\right)\right|^2\mathrm{d}t.\end{aligned} \tag{3.3}$$

Hence, from (3.1) by (3.3) the asymptotic formula (2.5) follows.

3.2. Let us remind the estimates (see [4], (3.15))

$$\varphi_1^k(T+U)-\varphi_1^k(T)<\frac{2k+1}{2n+1}\frac{T}{\ln T}\leq\frac{T}{\ln T},\ k=1,\dots,n.$$

It is clear that by the substitution

$$2n+1\to(2n+3)^2$$

(for example) in [4], part 3.4 we obtain the estimates

$$\varphi_1^k(T+U)-\varphi_1^k(T)<\frac{2k+1}{(2n+3)^2}\frac{T}{\ln T}<\frac{1}{2n+3}\frac{T}{\ln T},\ k=1,\dots,n+1, \tag{3.4}$$

i. e. the inequality (2.6) holds true.

Next we have (see [4], (3.4))

$$\varphi_1^k(T)-\varphi_1^{k+1}(T+U)+\varphi_1^{k+1}(T+U)-\varphi_1^{k+1}(T)>(1-\epsilon)(1-c)\frac{T}{\ln T}. \tag{3.5}$$

Consequently we have (see (3.4))

$$\begin{aligned}&\varphi_1^k(T)-\varphi_1^{k+1}(T+U)>(1-\epsilon)(1-c)\frac{T}{\ln T}-\{\varphi_1^{k+1}(T+U)-\varphi_1^{k+1}(T)\}> \\ &>(1-\epsilon)(1-c)\frac{T}{\ln T}-\frac{1}{2n+3}\frac{T}{\ln T}>\left(1-c-\frac{1}{2n+3}-\epsilon\right)\frac{T}{\ln T}\geq \\ &\geq\left(1-c-\frac{1}{5}-\epsilon\right)\frac{T}{\ln T}>(0.22-\epsilon)\frac{T}{\ln T}>0.2\frac{T}{\ln T},\end{aligned}$$

since

$$c<0.58\ \Rightarrow\ 1-c>0.42>\frac{1}{5}=0.2,$$

i. e. the inequality (2.7) holds true.

3.3. We use the Hardy-Littlewood-Ingham formula

$$\int_T^{T+U}Z^2(t)\mathrm{d}t\sim U\ln T,\ U\in\left[T^{\frac{1}{3}+\epsilon},\frac{T}{\ln^2T}\right] \tag{3.6}$$

where $\frac{1}{3}$ is the Balasubramanian exponent (the small improvements of the exponent $\frac{1}{3}$ are irrelevant for our purpose), and our formula (see [3], (2.5))

$$\int_T^{T+U}Z^2(t)\mathrm{d}t\sim\{\varphi_1(T+U)-\varphi_1(T)\}\ln T. \tag{3.7}$$

Comparing the formulae (3.6) and (3.7) we obtain

$$\varphi_1^1(T+U)-\varphi_1^1(T)=\varphi_1(T+U)-\varphi_1(T)\sim U.$$

Similarly, by comparison in the cases (see (2.6))

$$T\to\varphi_1^1(T),\ T+U\to\varphi_1^1(T+U);\dots$$

we obtain

$$\varphi_1^k(T+U)-\varphi_1^k(T)\sim U,\ k=1,\dots,n+1 \tag{3.8}$$

i. e. the formula (2.9) holds true.

3.4. Next, we have (see (2.4), (3.5), (3.8))

$$\varphi_1^k(T)-\varphi_1^{k+1}(T+U)+\varphi_1^{k+1}(T+U)-\varphi_1^{k+1}(T)\sim(1-c)\frac{T}{\ln T},$$

$$\varphi_1^k(T)-\varphi_1^{k+1}(T+U)\sim(1-c)\frac{T}{\ln T}-\{\varphi_1^{k+1}(T+U)-\varphi_1^{k+1}(T)\}\sim$$

$$\sim(1-c)\frac{T}{\ln T}-U\sim(1-c)\frac{T}{\ln T}$$

i. e. the formula (2.10) holds true. The proposition (2.11) follows from (2.10) .

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